STATISTICS OF CLASS GROUPS

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ABSTRACT. This project concerns statistics of class groups of imaginary quadratic fields, many features of which are accurately predicted by a probabilistic model proposed by Cohen and Lenstra. For instance, Soundararajan considered the number $\mathscr{F}(h)$ of imaginary quadratic fields with class number h, and asked how this value changes as h increases without bound. He conjectured an order of magnitude for $\mathscr{F}(h)$ as h approaches infinity, and this was refined in the work of Holmin, Jones, Kurlberg, McLeman and Petersen, who used Cohen-Lenstra Heuristics to conjecture a precise asymptotic formula for $\mathscr{F}(h)$ as h approaches infinity through odd values. This project aims to formulate a similar conjecture for even values of h, wherein the influence of Cohen-Lenstra becomes entangled with classical genus theory. Pursuant to this goal, the problem naturally arises of directly extending the Cohen-Lenstra Heuristics to groups of even order, which was partially carried out by Gerth, and our work extends his ideas. We formulate a conjecture for the asymptotic proportion of imaginary quadratic fields for which the two part of the class group is isomorphic to a fixed abelian two-group. We collect data supporting our conjecture, which reproduces data seen in a table of Mark Watkins. Our methods involve analytic and algebraic techniques and makes use of the free, open-source mathematics software SageMath. In future work we hope to complete the task of formulating a conjecture on the asymptotic nature of $\mathscr{F}(h)$ as h goes to infinity through even values.

1. INTRODUCTION

This project considers the relative distribution of the 2-part of the class group of imaginary quadratic fields, building on the work of Cohen-Lenstra (as extended by Gerth) and with the aim of refining a conjecture of Soundararajan on the number of imaginary quadratic fields with a given class number.

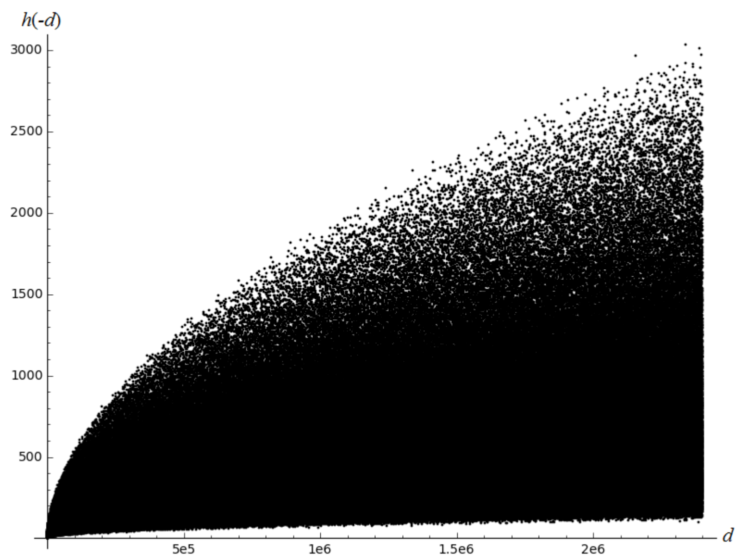


Figure 1. $h(-d)$ for $d \leq 2.4 \cdot 10^6$

The class group $Cl(\mathbb{Q}(\sqrt{-d}))$ of an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ is a finite abelian group which provides a measure of the failure of unique factorization in the ring of integers of $\mathbb{Q}(\sqrt{-d})$. The size of the class group is called the class number and is denoted by $h(-d)$. Class numbers and class groups have been studied by mathematicians for centuries and yet many of their basic properties continue to remain mysterious. For an *odd* prime p, Cohen and Lenstra developed heuristics predicting the likelihood that $Cl(\mathbb{Q}(\sqrt{-d}))_p \simeq A$, where A is a fixed abelian p-group and $Cl(\mathbb{Q}(\sqrt{-d}))_p$ denotes the p-part of the class group of $\mathbb{Q}(\sqrt{-d})$. When $p = 2$, the situation is somewhat complicated by classical genus theory, but Gerth was nevertheless able to find appropriate extensions of the Cohen-Lenstra heuristics to this case as well.

Our project aims to understand the count of imaginary quadratic fields having a prescribed class number. That is, we want to learn more about the growth as $h \to \infty$ of

$$\mathscr{F}(h) := \#\{\mathbb{Q}(\sqrt{-d}) : h(-d) = h\},$$

a quantity first introduced by Soundararajan. He established an asymptotic formula for the average value of $\mathscr{F}(h)$ and also conjectured that

$$\mathscr{F}(h) \asymp \frac{h}{\log(h)} \sum_{1 \le t \le v_2(h)+1} \frac{2^{t-1}(\log\log h)^{t-1}}{(t-1)!}. \tag{1}$$

In particular, for h odd, Soundararajan's conjecture reads

$$\mathscr{F}(h) \asymp \frac{h}{\log(h)} \qquad (h \text{ odd}).$$

Watkins first computed $F(h)$ for $h \le 100$ as shown in his table below. Notice that $\mathscr{F}(h)$ is relatively large when $2|h$.

Watkins' Table

h	$\mathscr{F}(h)$	**largest** d	h	$\mathscr{F}(h)$	**largest** d	h	$\mathscr{F}(h)$	**largest** d	h	$\mathscr{F}(h)$	**largest** d
1	9	63	26	190	103027	51	159	546067	76	1075	1086187
2	18	427	27	93	103387	52	770	439147	77	216	1242763
3	16	907	28	457	126043	53	114	425107	78	561	1004347
4	54	1555	29	83	166147	54	427	532123	79	175	1333963
5	25	2683	30	255	134467	55	163	452083	80	2277	1165483
6	51	3763	31	73	133387	56	1205	494323	81	228	1030723
7	31	5923	32	708	164803	57	179	615883	82	402	1446547
8	131	6307	33	101	222643	58	291	586987	83	150	1074907
9	34	10627	34	219	189883	59	128	474307	84	1715	1225387
10	87	13843	35	103	210907	60	1302	662803	85	221	1285747
11	41	15667	36	668	217627	61	132	606643	86	472	1534723
12	206	17803	37	85	158923	62	323	647707	87	222	1261747
13	37	20563	38	237	289963	63	216	991027	88	1905	1265587
14	95	30067	39	115	253507	64	1672	693067	89	192	1429387
15	68	34483	40	912	260947	65	164	703123	90	801	1548523
16	322	31243	41	109	296587	66	530	958483	91	214	1391083
17	45	37123	42	339	280267	67	120	652723	92	1248	1452067
18	150	48427	43	106	300787	68	976	819163	93	262	1475203
19	47	38707	44	691	319867	69	209	888427	94	509	1587763
20	350	58507	45	154	308323	70	560	811507	95	241	1659067
21	85	61483	46	268	462883	71	150	909547	96	3283	1684027
22	139	85507	47	107	375523	72	1930	947923	97	185	1842523
23	68	90787	48	1365	335203	73	119	886867	98	580	2383747
24	511	111763	49	132	393187	74	407	951043	99	289	1480627
25	95	93307	50	345	389467	75	237	916507	100	1763	1856563

The colors indicate the highest power of 2 dividing h. **largest** d is the largest d for which $h(-d) = h$.

In more recent work, Holmin, Jones, Kurlberg, McLeman, Peterson used divisibility statistics coming from the Cohen-Lenstra heuristics to refine this order of magnitude to a conjectural asymptotic formula:

$$F(h) \sim \mathscr{C} \cdot c(h) \cdot \frac{h}{\log(\pi h)} \qquad (h \text{ odd}), \tag{2}$$

where

$$\mathscr{C} := 15 \prod_{\substack{\ell \text{ prime} \\ \ell \geq 3}}^{\infty} \prod_{i=2}^{\infty} \left(1 - \frac{1}{\ell^i}\right) \approx 11.317$$

and

$$c(h) := \prod_{p^n || h} \prod_{i=1}^{n} \left(1 - \frac{1}{p^i}\right)^{-1}$$

The eventual goal of this project is to extend (2) to the case $h = 2^k m$ for m odd. As mentioned earlier, since the prime $p = 2$ is involved, the picture is complicated by classical genus theory, which dictates that

$$\text{rank}_2(Cl(\mathbb{Q}(\sqrt{-d}))) = t - 1 \;\Leftrightarrow\; \omega(d) = t.$$

We are led to consider the limiting density $\lim_{x\to\infty} L_t(G,x)$, where G is a fixed finite abelain 2-group,

$$L_t(G,x) := \frac{\#\{d \in D_t(x) : Cl(\mathbb{Q}(\sqrt{-d}))_2 \simeq G\}}{\#D_t(x)}$$

and

$$D_t(x) := \{d \leq x : -d \text{ is a fundamental discriminant and } \omega(d) = t\}.$$

Gerth's extension of Cohen-Lenstra Heuristics to the $p = 2$ case leads to the following prediction. Let e denote the 4-rank of G, and define

$$\mathscr{G}_e := \{\text{abelian groups } H : \ \text{rank}_2(H) = e\}$$
$$\mu_e(H) := \left(\frac{1}{\#\text{Aut}(H)}\right) / \left(\sum_{H' \in \mathscr{G}_e} \frac{1}{\#\text{Aut}(H')}\right).$$

Our analysis leads to the following conjecture. In its statement, the quantity

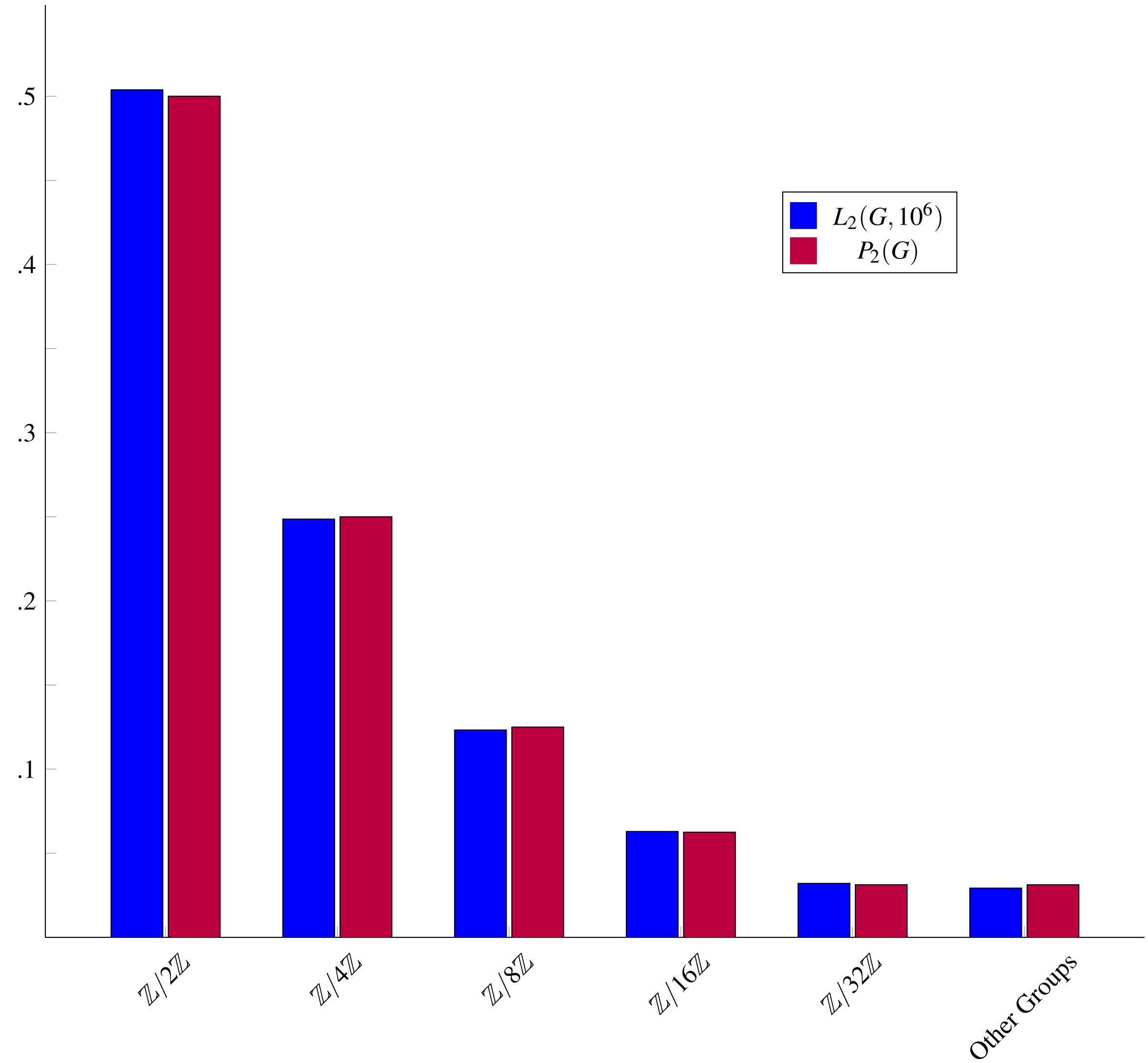
$$d_{t,e} = \sum_{\substack{1 \leq \ell \leq t \\ \ell \text{ odd}}} c_{t,\ell} f_{t,\ell,e}$$

where $c_{t,\ell} = \binom{t}{l} 2^{-(t-l)}$ and $f_{t,\ell,e}$ represents the probability that a randomly chosen matrix M with each entry $a_{ij} \in \mathbb{F}_2$ with $a_{ij} \neq a_{ji}$ when $1 \leq i \leq j \leq \ell - 1$ and with $a_{ij} = a_{ji}$ when $\ell \leq i \leq t-1$ and $1 \leq j \leq t-1$ has $rank(M) = t - 1 - e$.

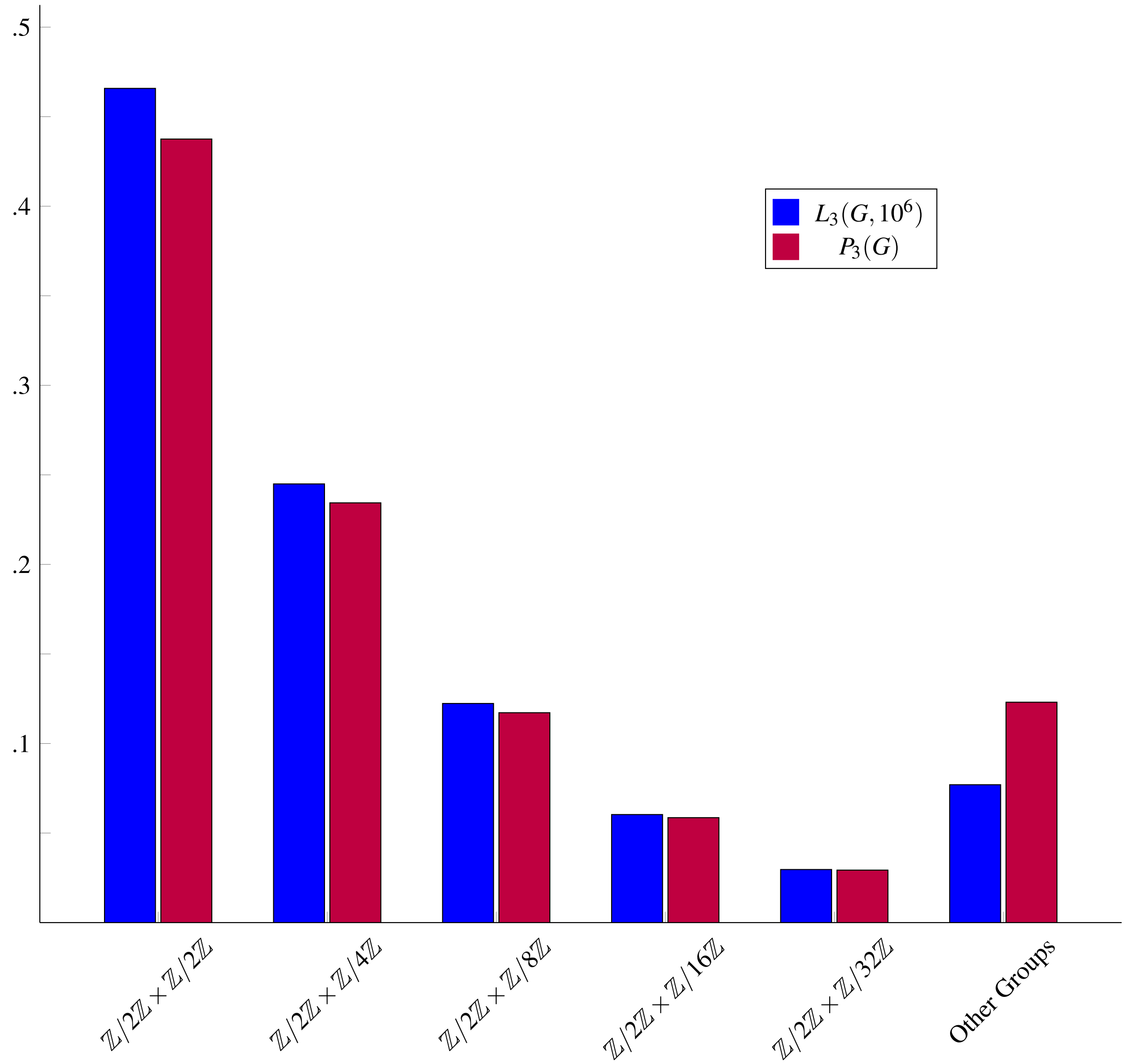
Conjecture 1. *One has that*

$$\lim_{x\to\infty} L_t(G,x) = d_{t,e} \cdot \mu_e(2G).$$

In future work, we plan to use this conjecture to refine (1) to an asymptotic formula for $\mathscr{F}(2^k m)$ as $m \to \infty$ through odd values, thus extending (2) to include the case of h even.



.5
.4
.3
.2
.1
$L_2(G, 10^6)$
$P_2(G)$
$\mathbb{Z}/2\mathbb{Z}$
$\mathbb{Z}/4\mathbb{Z}$
$\mathbb{Z}/8\mathbb{Z}$
$\mathbb{Z}/16\mathbb{Z}$
$\mathbb{Z}/32\mathbb{Z}$
Other Groups



References

[1] H. Cohen and H.W. Lenstra. Heuristics on class groups of number fields. Lecture Notes in Mathematics, **1068**, Springer, 1984, pp. 33–62.

[2] A. Granville and K. Soundararajan. The distribution of values of $L(1,\chi_d)$. *Geom. and Funct. Anal.* **13** (2003), 992–1028.

[3] S. Holmin, N. Jones, P. Kurlberg, S. McLeman and K. Petersen. Missing class groups and class number statistics for imaginary quadratic fields. Preprint, available at `http://arxiv.org/abs/1510.04387`

[4] K. Soundararajan. The number of imaginary quadratic fields with a given class number. *Hardy-Ramanujan J.* **30** (2007), 13–18.

[5] M. Watkins. Class numbers of imaginary quadratic fields. *Math. Comp.* **73** (2003), 907–938.