

Summary

A tangle toy is composed of arcs. Each arc is a quarter torus, whose end can be joined to another arc by a joint that holds the links together. At each joint, the joined arcs can rotate freely, changing the shape of the tangle toy. Describing all the possible shapes a Tangle Toy can take poses a fascinating challenge. For a tangle with more than eight links, the space of shapes it can take has a high dimension. However, with eight links, the dimension is small enough for the entire space of shapes to be visualized.

This tool, The Virtual Tangle Toy, is a program written in JavaScript that allows for user interaction with a virtual tangle toy. This program was created in a Fall 2016 semester project in the Mathematical Computing Laboratory at UIC with faculty advisor David Dumas.

Motivation

Tangle toys are plastic mechanical linkages that can be rotated to form various shapes. K. Rafi and G. Zhang studied the configuration space of the closed eight-link toy in [1].

Rafi-Zhang highlight seven symmetrical shapes in their research that inspired this research project. The focus of our research was to create a visualization companion to the theoretical results in Rafi-Zhang's paper. We were able to make a program that visualizes the Rafi-Zhang research on the eight link tangle toy. The program also lets users visualize tangle toys of varying link sizes, and lets user add up to 15 links.

Angle Measurements

We can measure the position of any joint in the tangle toy with a real-valued angle. Each arc is on a plane, and when two arcs are joined, the angle of rotation about the joint is the angle between the two planes.

An angle can take values in $[-\pi, \pi]$. If the angle of rotation is 0, then the two arcs form a half-circle.



Angles: 0, π , 0. Angles: 0,0,0,0.

Calculations

Creating The Virtual Tangle Toy consisted of many calculations including matrix multiplication. In this JavaScript + WebGL program, each arc is on a plane, and its position and orientation are represented by a 4x4 matrix.

The first arc's position is the Identity matrix I . The positioning and orientation of each other arc in the tangle toy is produced by matrix multiplication. Translation and rotation in the program is done with the use of the following matrix $NR =$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ \cos(t) & 0 & -\sin(t) & -\cos(t)+1 \\ \sin(t) & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If two arcs, a_1 and a_2 , meet with a joint angle t , then their associated matrices M_1 and M_2 satisfy:

$$M_2 = NR * M_1$$

The tangle toy is closed when the starting position and orientation of the first arc and the ending position and orientation of the last arc are equal. Mathematically, this happens if the product of the matrices NR (over all of the joints, in order) is equal to the identity matrix.

Building the Virtual Tangle Toy

The Virtual Tangle toy was built and visualized using JavaScript with its *three.js* and *dat.gui* libraries. The first steps in creating the program included building the arcs themselves. After building the arcs, the ability to add additional arcs and change joint angles using sliders were coded in. That was followed by the addition of the drop down menu to change the 8 link tangle toy to a specific shape.

Three.js provided the resources to create and manipulate a colored tangle toy and the *dat.gui* library provided the resources to support the ability to receive and respond to keyboard and mouse input [3].

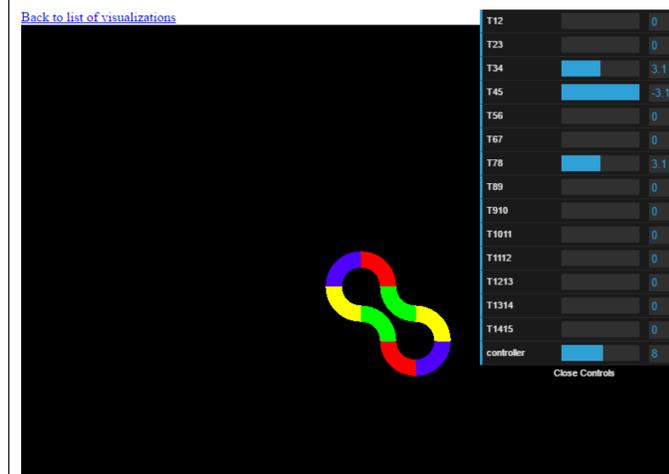
Sage Math, a Python based computational algebra system, was used in determining the angles of the 8 link symmetrical shapes. We performed matrix multiplication on Sage Math to help determine these angles. You can find Sage Math at: www.sagemath.org[2]

Future Objectives

The current program could be modified to create smooth transitions between the eight link symmetrical shapes. At the moment, it simply skips from one to the other. With a physical tangle toy, you can see how the tangle toy changes from one symmetrical shape to the next. The changes are gradual, and quite interesting. For example, many of the symmetrical shapes are very close to the peanut. While it is really simple to see this by manipulating a physical tangle toy, it is not that easy with the current version of the Virtual Tangle Toy.

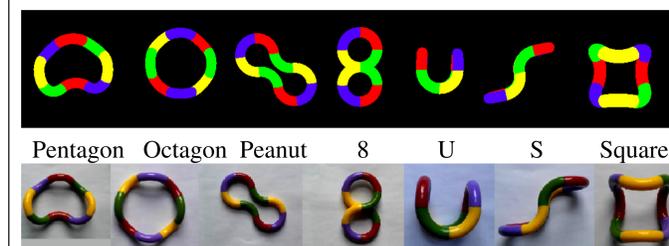
Using the Virtual Tangle Toy

The Virtual Tangle Toy allows users to examine a tangle toy with a computer. While a majority of the Rafi-Zhang research and the Virtual Tangle Toy focus on the eight link case of a tangle toy, users can freely add/remove links if they want. Currently, the limit of links that can be added is 15. When the Virtual Tangle Toy is initially run, eight joined links appear in the center of the screen. Users can adjust the number of links with the use of a slider. Users can also change the angle of rotation of each joint to see the possible shapes the tangle toy takes.



8 Link Tangle Toy

In the paper of Rafi-Zhang [1], seven symmetrical shapes of the 8-link tangle toy have a key role in mapping the configuration space. These are shown below as visualized with our program and with a physical tangle toy.



References

- 1: K. Rafi, G. Zhang. *Moduli Space of the Tangle Toy*. Preprint, 2015.
- 2: W. Stein. *Sage: Creating a Viable Free Open Source Alternative to Magma, Maple, Mathematica, and MATLAB*. Foundations of Computational Mathematics, Budapest 2011, pages 230-238, London Math. Soc. Lecture Note Ser., 403, Cambridge Univ. Press, Cambridge, 2013.
- 3: R. Cabello. *three.js*, 2010