The Idea of Unfolding

One of the main ideas, which Alicia Boole Stott used to visualize four-dimensional polytopes, is to imagine unfolding them and then studying the unfolded structure which is in one dimension lower.

This idea is familiar to every one of us, as we use it, for example, when we want to build a paper model of a 3-dimensional cube. Instead of looking at a cube which is a three-dimensional object we can unfold it to obtain six squares in two dimensions. By imagining how these squares will combine together to form a cube, that is, which edges and vertices of the squares will coincide, when we fold the cube, we can imagine how a cube is formed using these two-dimensional squares.

In Figure 1(a) we see a structure in the plane, formed by six squares. Each square has four edges, some edges are shared by two squares. When an edge is shared by two squares, we say that these two squares are identified along an edge. In Figure 1(a), each edge of the central square is identified with an edge in another square. Four of the squares have one edge identified with an edge in another square, and one square has two edges identified with edges in two other squares.

In Figure 1(b), we start folding the planar structure from Figure 1(a) into a cube. In Figure 1(c), the cube is partially folded. Four pairs of edges which belonged to different squares in Figure 1(a) became identified into four edges, each shared by two squares. To complete folding the cube, one has to identify three more pairs of edges. Each vertex in a three-dimensional cube is adjacent to three squares.

The Method of Sections and the Hypercube

Alicia Boole Stott’s method of understanding polytopes in four dimensions involved taking sections of them with 3d planes and then studying these sections.

For example, in a 3d cube we can take a section of the 2-dimensional plane S1 so that it intersects the cube along one of its faces, for example, along the base square in Figure 1(c). This square corresponds to the central square in Figure 1(a). Now take a plane parallel to S1, then it intersects four squares in the 3d cube along line segments, which form again a square. In Figure 1(c), this section is represented by a blue rubber band wrapped around the 3d cube.

A polytope in four dimensions, bounded by three-dimensional cubes is called a hypercube. Each vertex in the hypercube is adjacent to four 3-dimensional cubes. How many 3-dimensional cubes does one need to build a hypercube?

Figure 2 represents four cubes, adjacent to the same vertex in a hypercube, unfolded into the 3-dimensional space. Just like in Figure 1(a) a pair of edges in the unfolded cube represents the same edge in the folded cube. In Figure 2, a pair of squares in different 3d cubes may represent the same square in the unfolded hypercube.

To help understand, how cubes are folded together into a four-dimensional object, put vertices which fold into the same vertex are marked by the same color. For example, when the hypercube is folded, three vertices in Figure 2, marked by yellow color, are identified into the same vertex. Two vertices, marked by red color are folded into one vertex, two vertices, marked by purple color are folded into one vertex and so on.

Let $S_1$ be a three-dimensional hyperplane in the four-dimensional space, which intersects the hypercube along one of the bounding 3d cubes. In Figure 2, this is the central cube which is behind the three cubes which can be seen in the picture. Then take a hyperplane $S_2$ which is parallel to the hyperplane $S_1$, and suppose it intersects one of the cubes in Figure 2, except the central one. Then $S_2$ has to intersect this cube along a square, marked in Figure 2 by a red rubber band. Since the sides of cubes are identified, $S_2$ also has to intersect the other two cubes in Figure 2 along squares. These squares fold into three sides of a 3-dimensional cube. The remaining three squares of the intersection of $S_2$ and the hypercube are in the hypercube which are not represented in Figure 2.

These are the cubes attached to the central cube along the three edges not identified with other cubes in the picture. We conclude that the intersection of $S_2$ and the hypercube is a 3-dimensional cube.

Let $S_3$ be a three-dimensional hyperplane, parallel to $S_1$ and $S_2$, placed further away from the central cube. Then it intersects the hypercube along a 3-dimensional cube. The intersection is marked by blue rubber bands in Figure 2.