Vietoris-Rips complexes, random points on varieties, and persistent homology

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Summary

The project takes a variety of n dimensions intersected by random lines to generate points. From this point cloud, a weighted undirected neighborhood graph is constructed. Then, the neighborhood graph undergoes the Vietoris-Rips expansion by constructing cofaces for which the vertex is maximal.

Motivation

The goal of this project is to use persistent homology to study the topology of random complex algebraic varieties, geometric figures described as the solution sets of system of polynomial equations.

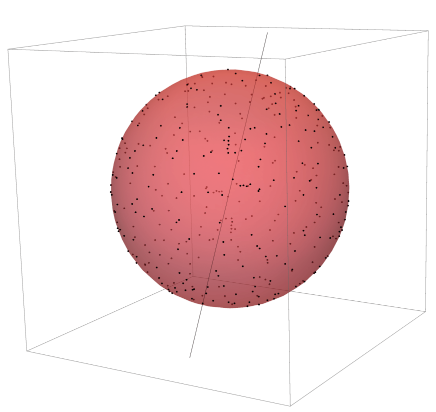
Using PHCpack to find points

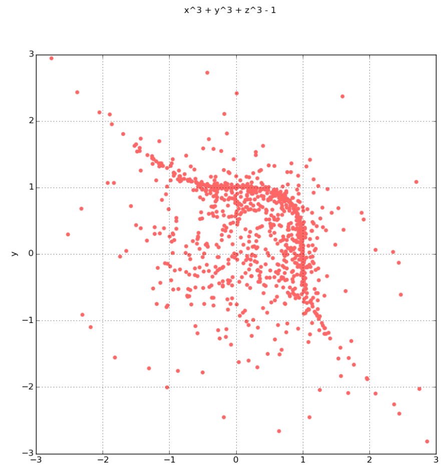
PHCpack is a polynomial system solver using homotopy continuation methods [1]. Let $f(x) = 0$ be the target system. PHC constructs $g(x) = 0$ start system. Then let γ be the accessibility constant, $\gamma \in \mathbb{C}$ and t be the continuation parameter, $t \in [0,1]$. The homotopy h is defined as

$$h(x,t) = \gamma \cdot (1-t) \cdot g(x) + t \cdot f(x) = 0$$

$$h(x,0) = g(x)$$

$$h(x,1) = f(x)$$





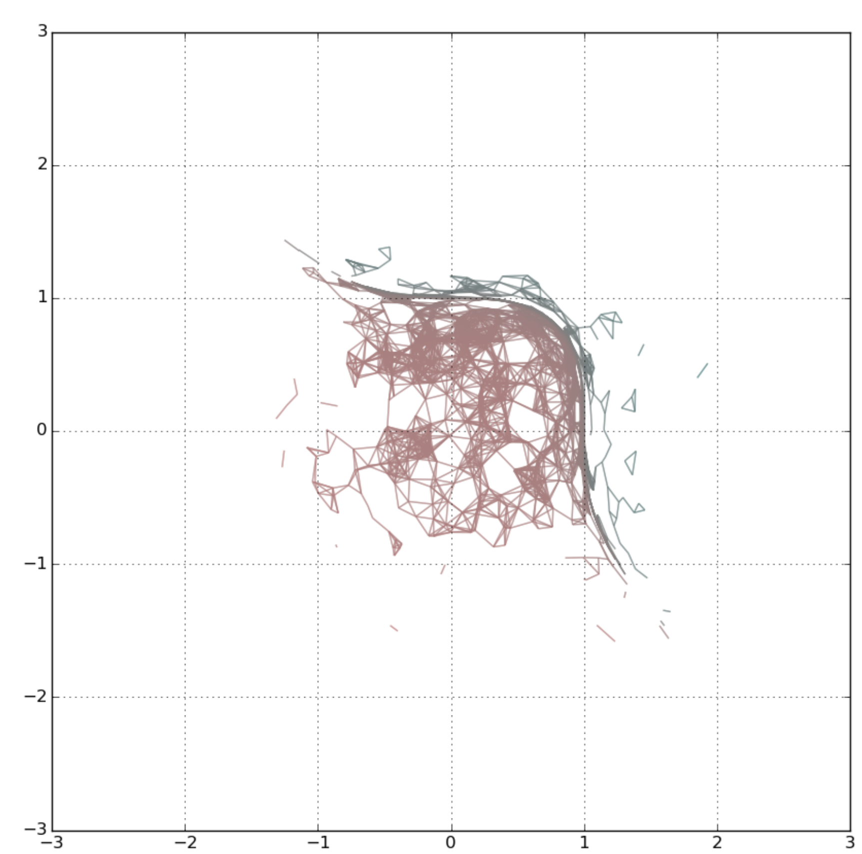
Citations

[1] Verschelde, J. (1999). "PHCpack: a general-purpose solver for polynomial systems by homotopy continuation." Algorithm 795 in ACM Trans. Math. Softw.

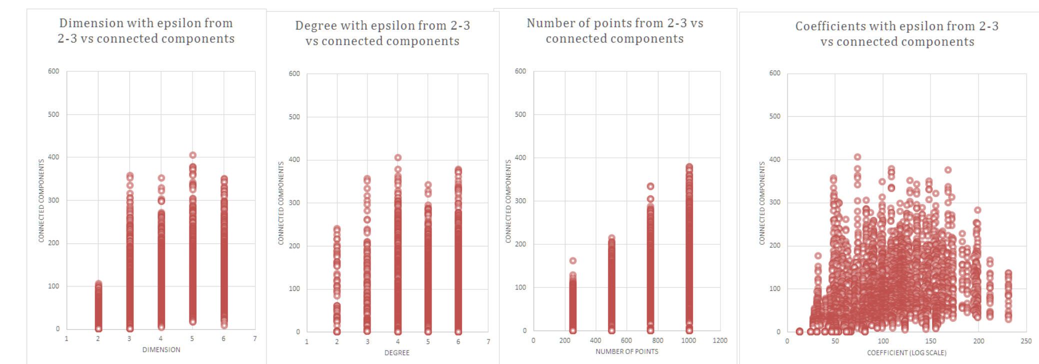
Neighborhood graphs

The neighborhood graph of a set S with a parameter ε is the graph with vertices in S, where an edge is contained in the neighborhood graph if and only if the distance between it's endpoints is less than ε. Our quick edge algorithm takes as input a set of points S of order n in affine space, and some positive real number ε, and produces a list of all edges that have length less than ε. Our method uses recursion. If our ambient space has dimension d, let k be some positive integer less than d.

First we compute the median value of the k-th coordinate of all the points in S, and call this m. Now we construct four subsets of S: A, B, A^*, and B^*. A is the set of points whose k-th coordinate is less than m. B is defined as $S - A$. A^* and B^* are subsets of A and B respectively where points in these sets have kth coordinate within ε of m. We add an edge to our list for every pair (a,b) where a in A^* and b in B^*, and they are within ε of each other. We then call the algorithm recursively, replacing S with A and B, each half the size.



Some connectivity tests



Persistence and future work

In the future, we would like to conduct a more thorough test relating connectivity, number of points, and the size of coefficients. In particular, we are interested in whether the sharp-threshold phenomenon found in random graphs can be detected in random points of algebraic varieties.

Another pretty picture

Below we show the neighborhood graph with $\varepsilon = .2$ associated to 5000 random points on the torus. In this picture there are 17 connected components.