Persistent Homology on Algebraic Varieties

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Objectives

1. Create a set of tools for analyzing the persistent homology of weighted simplicial complexes.
2. Apply these to points sampled on algebraic varieties.
3. Decide whether these techniques are fast enough to be useful in practice.

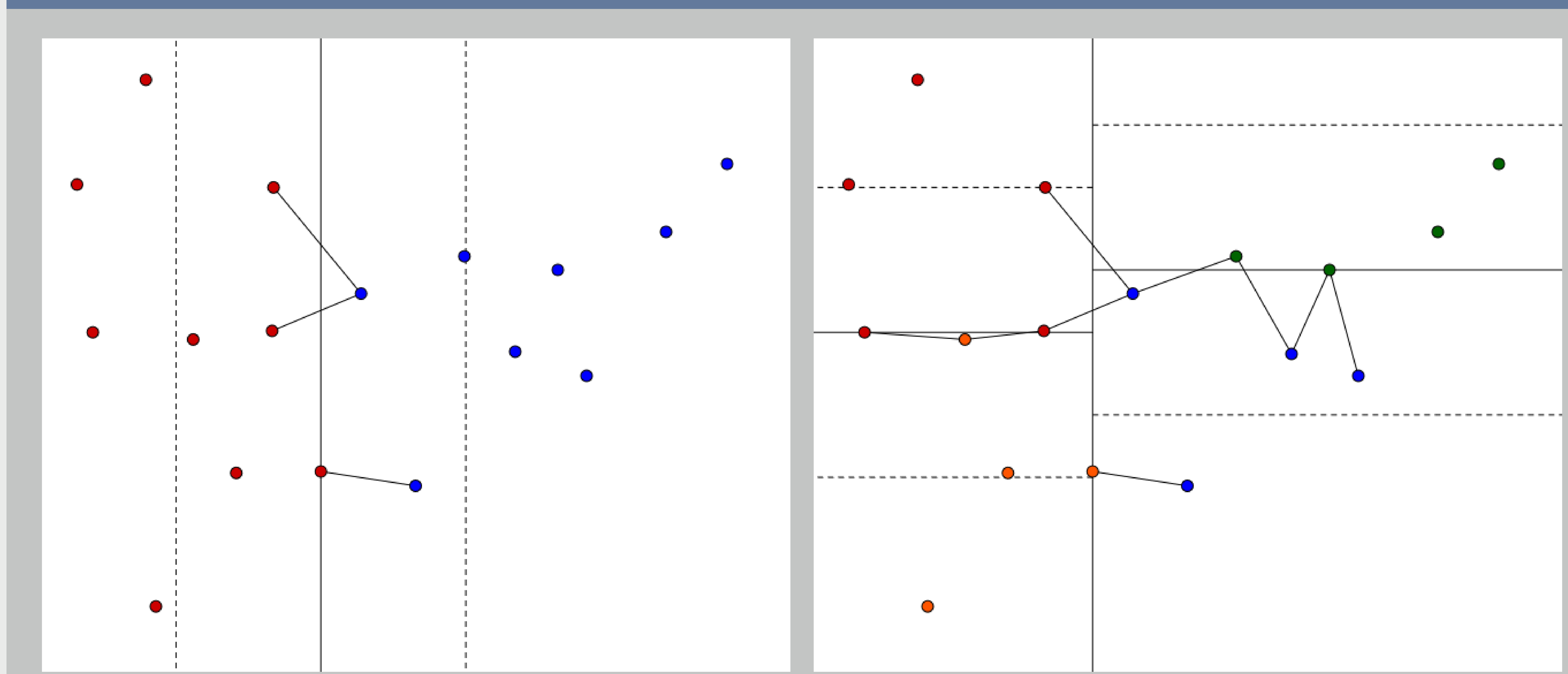
Introduction

- Persistent Homology is a method for computing the "shape" of a surface from a series of approximations of the surface.
- Multi-step process
  1. Sample points on a surface.
  2. Compute the neighborhood graph of these points.
  3. Compute the weighted simplicial complex containing these edges.
  4. Compute the persistent homology from the weighted simplicial complex.
  5. Draw the barcode graph.

Neighborhood Graph

- Given some value ϵ we construct the neighborhood graph of a set of points by including an edge if and only if the distance between its endpoints is less than or equal to ϵ.
  - can easily be restricted to $\epsilon' < \epsilon$
  - represents the data in a more discrete way.

QuickEdge for Computing Neighborhood Graphs



- The median of the x coordinates is found, and used to split the points into two sets. (shown as red and blue)
- The two dotted lines as shown are constructed to be ϵ apart.
- Two points are connected if they are not the same color, they are within the two dotted lines, and they are within epsilon of each other.
  - If two points are connected, add the edge.
  - Points outside the dotted lines need not be considered.
- Repeat this algorithm on the blue and red points recursively.

Weighted Simplicial Complexes

- A simplicial complex is a set of simplices which intersect along their sub-simplices.
- In a weighted simplicial complex, the simplices have weights.
  - We will require that all sub-simplices are also contained in the complex.
  - A simplex's weight is the maximum distance between two of its vertices.

Order on Simplices

- Simplices are sorted first by weight, then by degree, then by lexicographic ordering of the vertices (according to an arbitrary but consistent order).
- This represents the order the simplices are "born".

Persistent Homology

- Persistent Homology is a tool for approximating the Homology of a surface at different resolutions.
- elements that persist for longer durations are more likely to represent a real element of the underlying surface described by our algebraic variety.
  - These will be longer edges in the barcode graph.

Persistence Algorithm

- Construct an $n \times n$ matrix of elements of $\mathbb{Z}/2\mathbb{Z}$, where n is the number of simplices in the weighted simplicial complex.
  - we will represent this as a sparse matrix, so this will run much better than if it were represented discretely.
- the element at (x, y) is nonzero if the corresponding simplices σ_x, and σ_y have the property that σ_y is a face of σ_x.
- Let $low(j)$ be the index of the lowest nonzero element of the j^{th} column.
- If two rows have the same low value, then replace whichever has larger column index with the difference of the rows.
- Once no two rows have the same low value, two simplices form a persistence pair if $i = low(j)$ and $\forall k, (i, k) = 0$

Persistence Example

Consider the cube whose vertices are $\{0, 1\}^3$

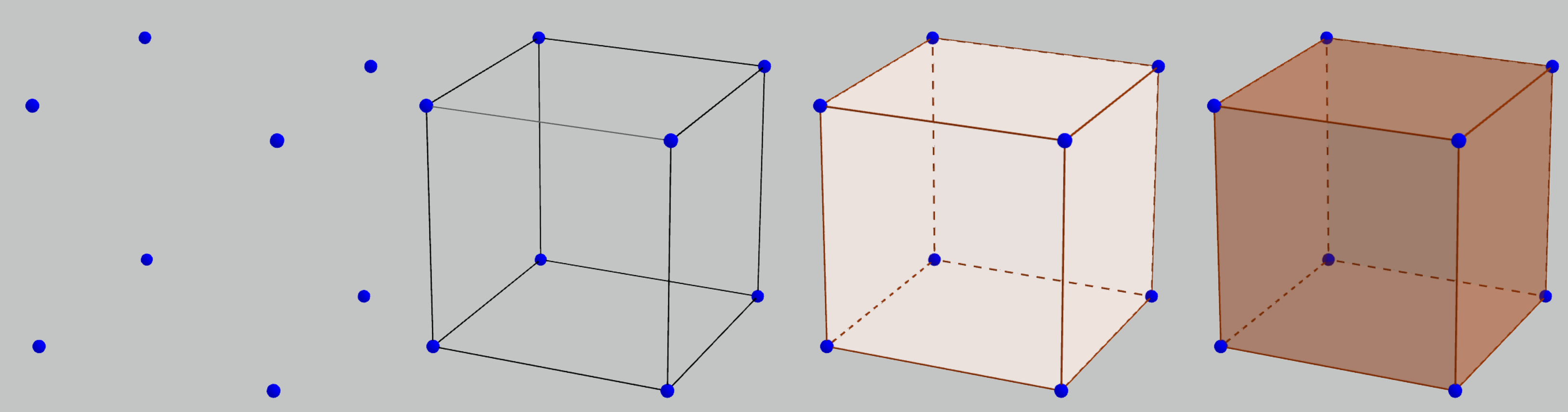


Figure 1: the k-skeletons of the cube

Running the persistence algorithm gives us the following barcode graph:

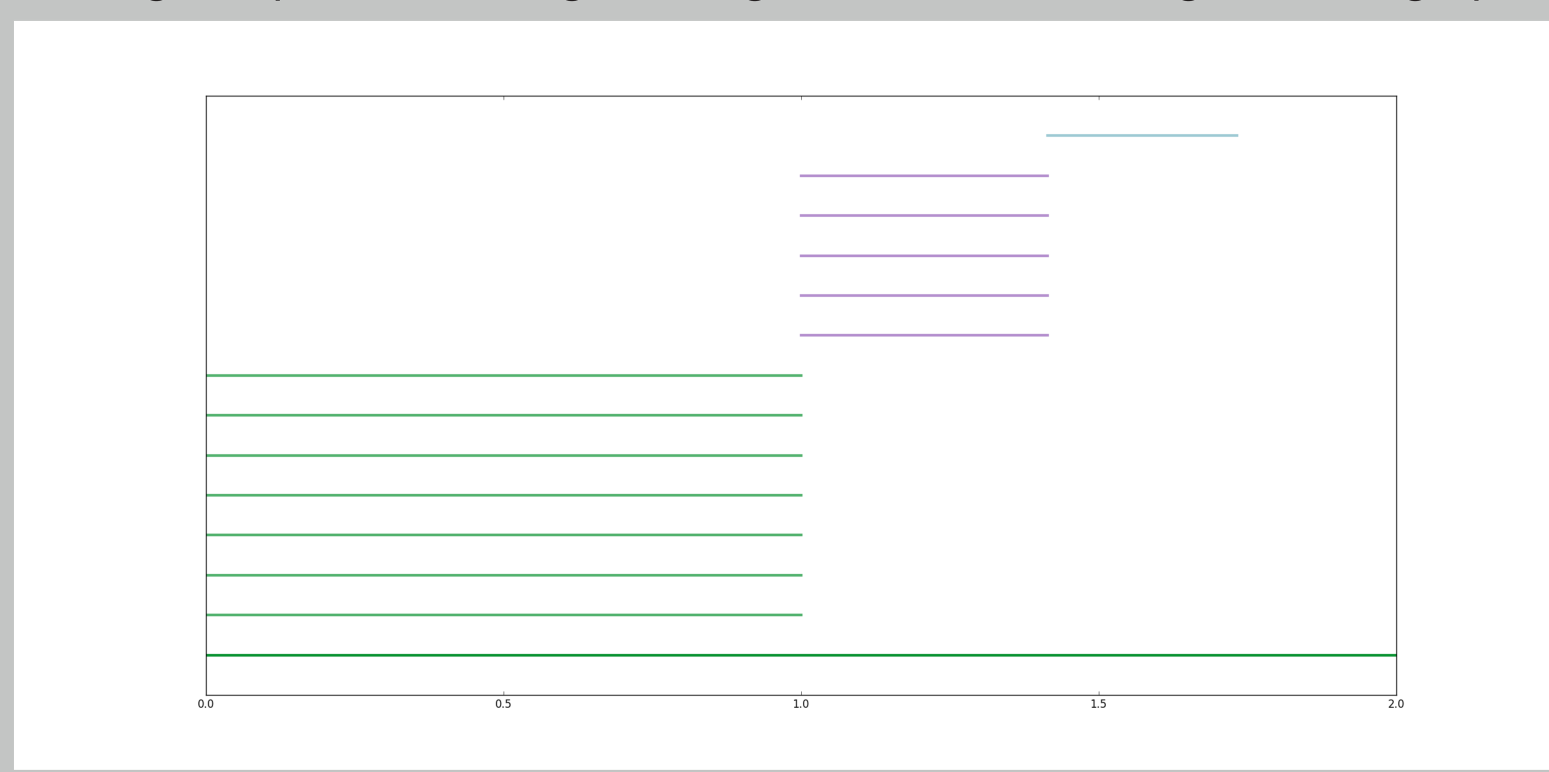


Figure 2: the barcode graph of the cube

Conclusion

- The problem with our method as of now is the speed. It can only handle complete graphs up to around 16 vertices until it becomes unreasonably long.
  - We can run our algorithm on graphs limited to simplices of a certain degree, but then the running time is still $O(n^d)$, which is fairly slow.

Acknowledgments

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