Brownian Motion on Manifolds

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Introduction

In 1827, botanist Robert Brown, through a microscope, observed something curious about the motion of tiny particles suspended in water: the particles, having been ejected from their pollen grain host, moved in a strangely jagged way. Such work attracted the attention of many, and inspired interest in developing a mathematical construct of their behavior.

In this project, we studied the properties of Brownian motion on Riemannian manifolds. In particular, we simulated several realizations of the Wiener process (the name for the mathematical construct of the Brownian motion) on the surface of a sphere, in order to examine Birkhoff's Ergodic Theorem.

Brownian Motion

A Brownian motion X_t on $t \geq 0$ is continuous-time stochastic process with the following characteristics:

1. *Independent increments.* $X_{t+s} - X_t$ is independent of $\sigma\{X_u | u \leq t\}$ for $s \geq 0$.
2. *Gaussian increments.* $X_{t+s} - X_t \sim N(0, s)$.
3. *Continuous paths.* With probability 1, X_t is continuous with respect to t.

Birkhoff's Ergodic Theorem

In the context of our project, Birkhoff's Ergodic Theorem tells us the following: For any real-valued measurable function $f : \mathcal{M} \to \mathbb{R}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s) ds \to \int_{\mathcal{M}} f(x) \mu(dx)$$

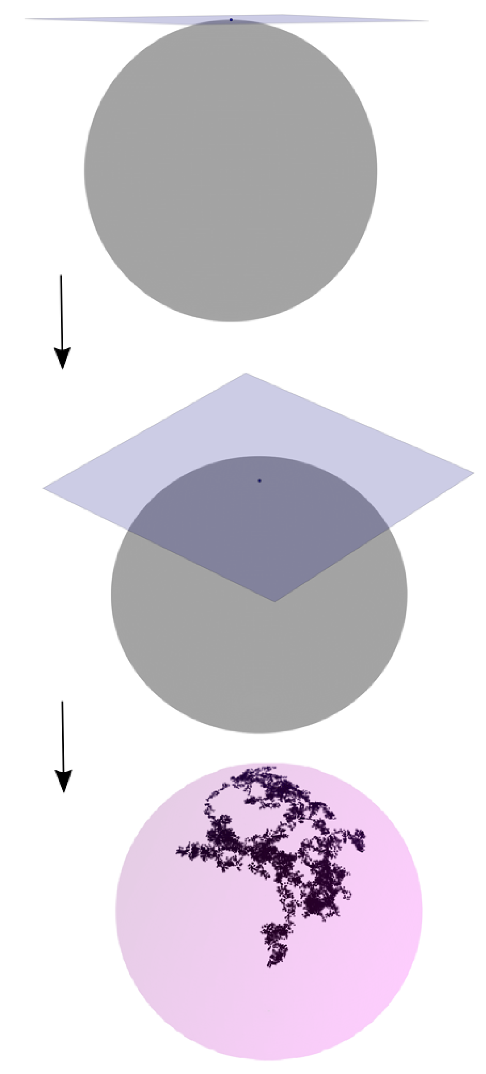
for any compact manifold $\mathcal{M}$ and μ is the normalized volume measure.

Approach

We simulated in both R and Python several Brownian motions on the surface of a sphere in two ways. The following outlines the approach used for the simulations:

1. On the plane tangent to the point $(0, r, 0)$ (referred to as "North pole") on a sphere centered at the origin, generate a single step of a Brownian motion starting at the point at which the plane and the sphere are in contact.
2. "Smooth" the step onto the sphere in such a way that the length of the step is preserved.
3. Rotate the sphere so that the end of the step is positioned at the North pole.
4. Repeat 1-3 until the desired number of steps is achieved.

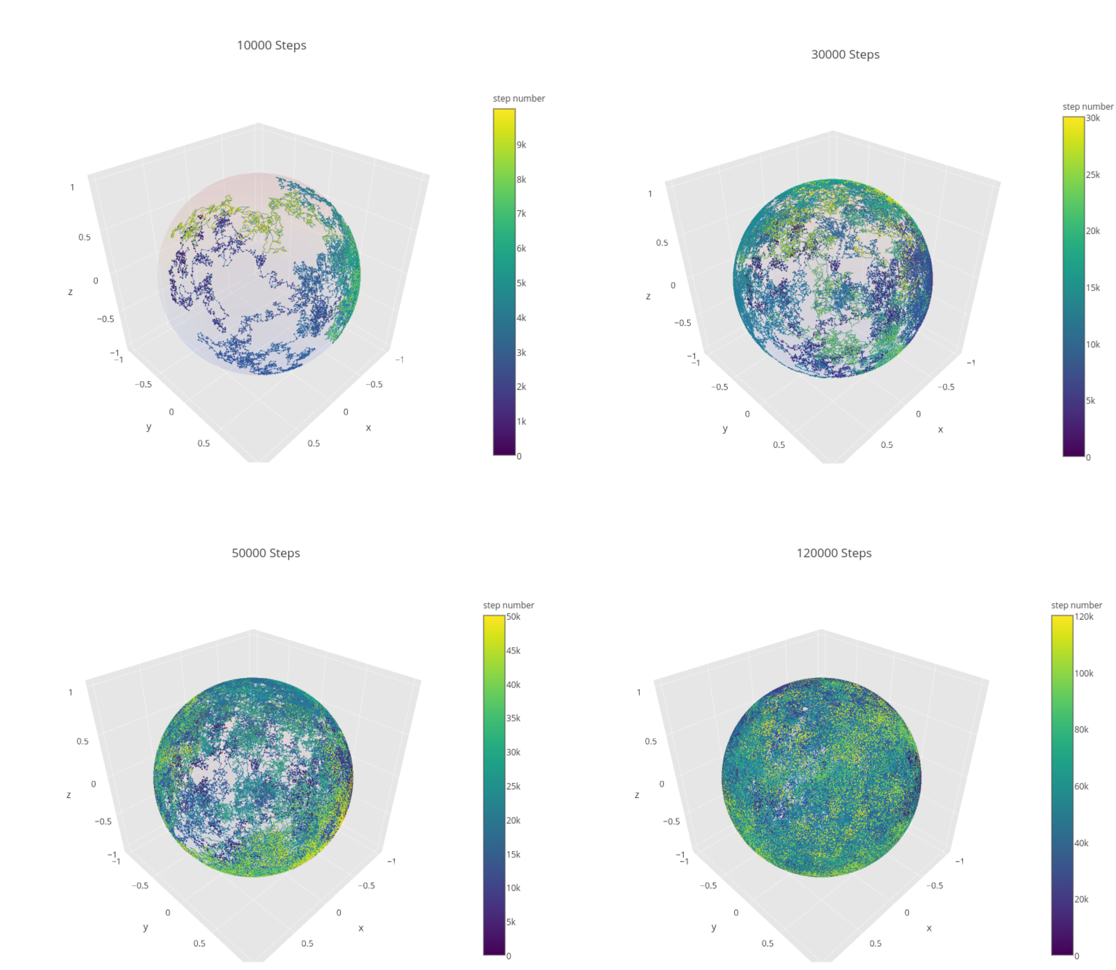
Visualizing the Approach



(Top) A walk is generated in the tangent plane at the pole, drawing Gaussian random numbers in $\mathbb{R}^2$. The particle is then projected onto the sphere.
(Middle) This process is then repeated in the tangent plane at the particle's new position on the surface of the sphere.
(Bottom) Many steps generated using this method.

A Simulation of Brownian Motion on S^2

- Radius of Sphere: 1
- Speed Parameter (step size): $0.0008\overline{3}$

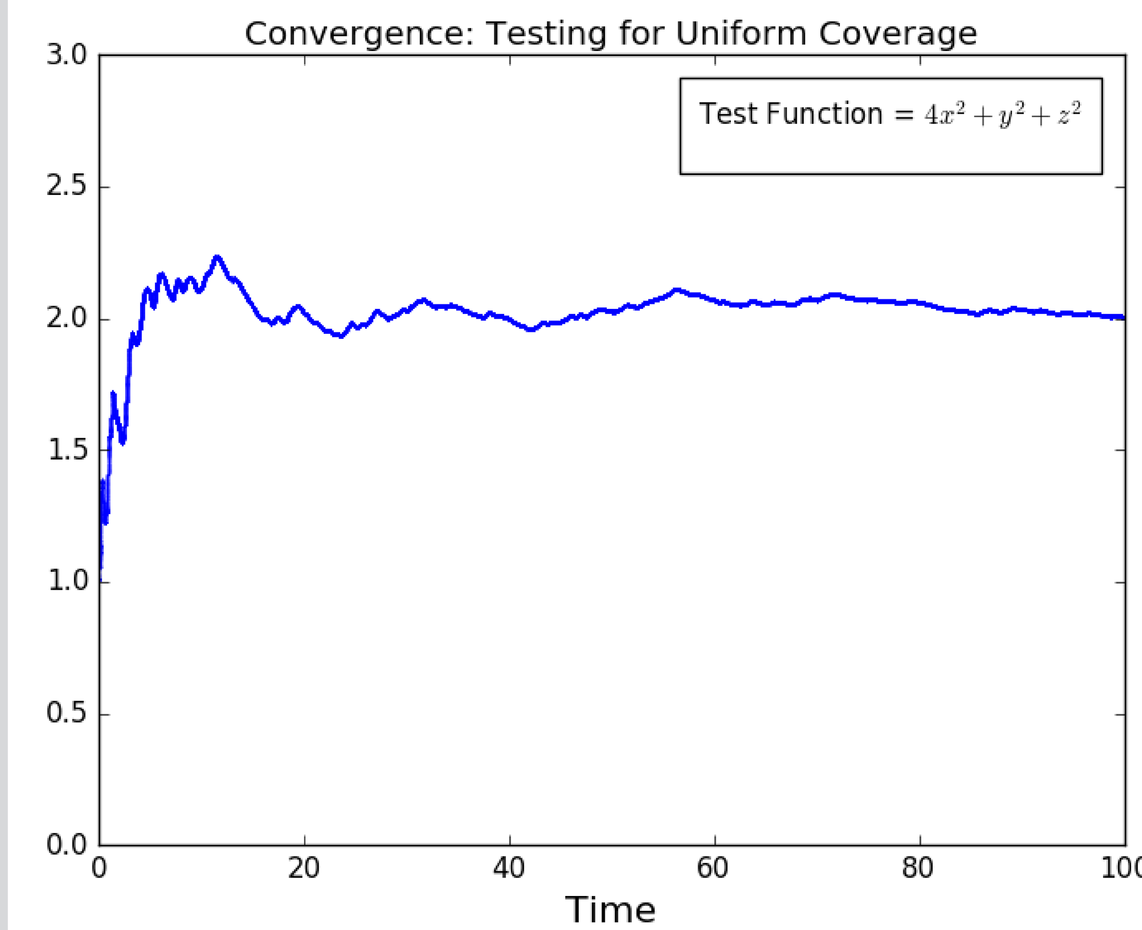


Results from the simulation after 10000, 30000, 50000, and 120000 steps

Results

Two remarks:

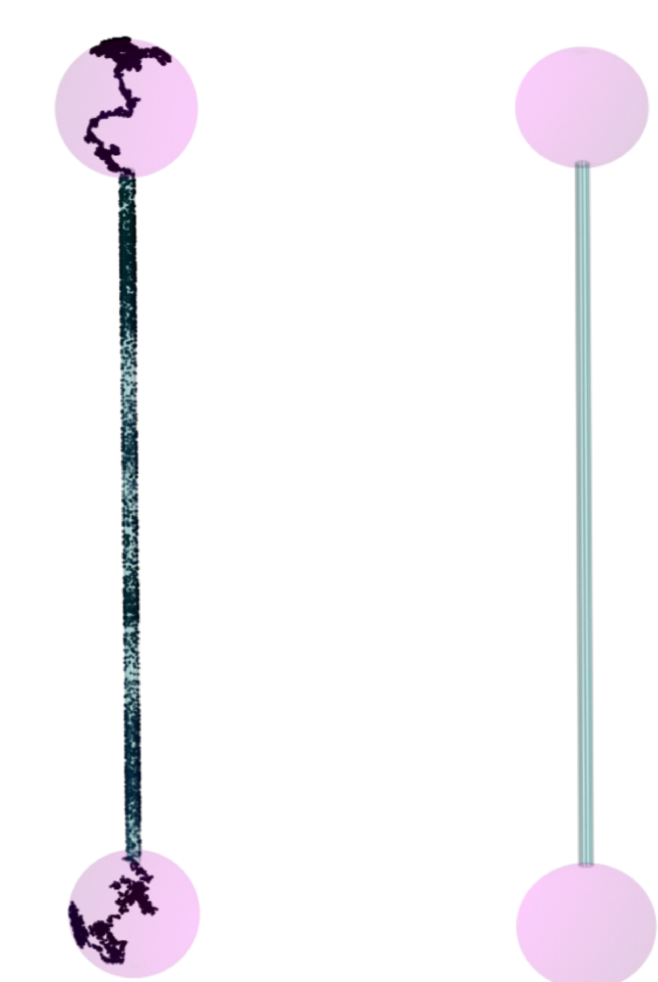
- The images to the bottom-left (Brownian Motion on S^2) show that after a sufficient amount of time, the Brownian motion will have visited on every part of the sphere uniformly.
- The graph below also verifies that, after a sufficient amount of time, the time-average of the Brownian motion's trajectory over f converges to the space average.



Brownian Motion as a Diffusion Process

Initial Simulation: 2 spheres connected at ends of a cylinder

- (Left) Number of steps combined across objects: 30000



Future Research

Future study branching from this project might have such foci as studying the convergence of a Brownian motion in a diffusion scenario or the convergence of a Brownian motion on Gabriel's Horn—an object whose volume is finite but whose surface area is infinite (will the motion become "trapped" near the mouthpiece?).

In fact, effort has already been made toward the former of such foci. Considering an object composed of two spheres connected by a bridge whose radius tapers from the sphere-bridge connection to the center of the bridge, we ask: in what way does the rate of convergence of the motion change when 1) the length of the bridge or 2) the opening of the sphere-bridge connection is modified?

It would also be convenient to convert the code used to run the simulations for this project into a tool for studying and/or continuing to develop methods relating to Brownian motion on manifolds.

Interact with the Manifolds

Visit our Plotly page at: `https://plot.ly/~besser2/`

References

L. Rey-Bellet, Ergodic properties of Markov processes, In *Open Quantum Systems II*, S. Attal, A Joye, C.-A. Pillet (Eds.), volume 1881 of *Lecture Notes in Mathematics* (Springer, 2006) pp. 1-39.