Visualizing Polytopes in Four Dimensions: the Work of Alicia Boole Stott

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Summary

The goal of our project was to understand the method of visualizing four-dimensional polytopes, using cross-sections. This method was developed in the late 19th - early 20th century by Alicia Boole Stott, an amateur mathematician who lived in Ireland and Britain.

There are six regular polytopes in four dimensions: the hypertetrahedron or the 5-cell, the hypercube or the 8-cell, the hyperoctahedron or the 16-cell, the 24-cell, the 120-cell and the 600-cell. Here the number, for example 8 in the name the '8-cell', refers to the number of three-dimensional polyhedra which form the faces of the four-dimensional polytope. Alicia Boole Stott's method consists of intersecting a four-dimensional polytope by a series of three-dimensional parallel planes. The intersection of the polytope and a plane is a three dimensional polyhedron.

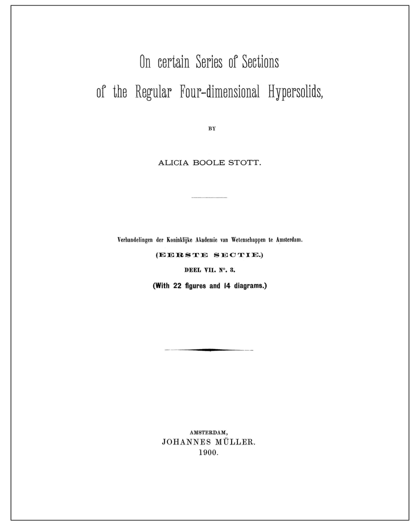
Alicia Boole Stott built series of paper models of the sections of all six four-dimensional polytopes. In our project, we studied the sections of the 5-cell and the 8-cell, and we printed the models and the sections for the 8-cell using a 3D printer.

Biography of Alicia Boole Scott

Alicia Boole Scott was born in 1860 in Cork, Ireland. She was a daughter of logician George Boole and Mary Everest Boole, and she had four sisters. George Boole died when Alicia was four years old. After his death, Alicia's mother and her sisters moved to London, where her mother was offered a job as a librarian at Queen's College. Alicia stayed in Ireland with her grandmother for a while, but eventually she joined her mother in London at the age of eleven.

Alicia did not receive any formal education, since there was not many educational opportunities for women at that time, and the family had financial difficulties. Her mother, however, learned mathematics when studying with her husband, George Boole, and she was interested in mathematical education. She wrote a few books about teaching children mathematics, which were published much later. It is possible that she tutored her daughter Alicia in mathematics.

During her time in London, Alicia also met the amateur mathematician Howard Hinton, who was a school teacher interested in two- and four-dimensional geometry. During this period Boole Stott started studying three- and four-dimensional geometry and contributed to Hinton's books. Alicia's method of visualizing four dimensional polytopes consisted of studying the series of their three-dimensional sections. She re-discovered the six regular polytopes, and build cardboard models of the series of sections. About 1895, she learned about the work of the Dutch mathematician P. H. Schoute, who studied the three-dimensional sections of four-dimensional polytopes by analytical methods. She wrote a letter to Schoute, describing her geometric results, and this started a collaboration which went on for about 20 years until the death of Schoute in 1913. Schoute encouraged Boole Stott to publish her results, and they also wrote a few papers together. To recognize Alicia Boole Stott's contribution to mathematics, the University of Groningen in The Netherlands, where P. H. Schoute was a professor of mathematics, awarded her an honorary doctorate in 1914.



On certain Series of Sections
of the Regular Four-dimensional Hypersolids,

BY

ALICIA BOOLE STOTT.

Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam.
(EERSTE SECTIE.)
DEEL VII. N°. 3.
(With 22 figures and 14 diagrams.)

AMSTERDAM,
JOHANNES MÜLLER.
1900.

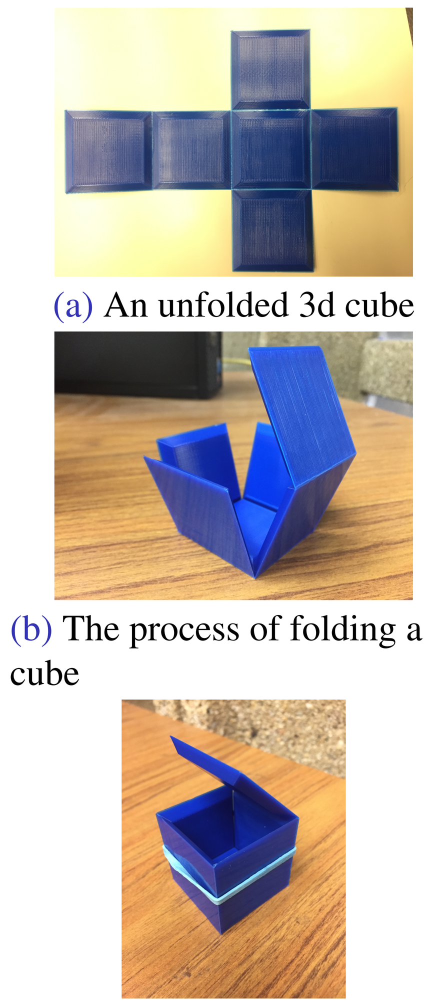
The Idea of Unfolding

One of the main ideas, which Alicia Boole Stott used to visualize four-dimensional polytopes was to imagine unfolding them and then studying the unfolded structure which is in one dimension lower.

This idea is familiar to every one of us, as we use it, for example, when we want to build a paper model of a 3-dimensional cube. Instead of looking at a cube which is a three-dimensional object we can unfold it to obtain six squares in two dimensions. By imagining how these squares will combine together to form a cube, that is, which edges and vertices of the squares will coincide, when we fold the cube, we can imagine how a cube is formed using these two dimensional squares.

In Figure 1(a) we see a structure in the plane, formed by six squares. Each square has four edges, some edges are shared by two squares. When an edge is shared by two squares, we say that these two squares are *identified along an edge*. In Figure 1(a), each edge of the central square is identified with an edge in another square. Four of the squares have one edge identified with an edge in another square, and one square has two edges identified with edges in two other squares.

In Figure 1(b), we start folding the planar structure from Figure 1(a) into a cube. In Figure 1(c), the cube is partially folded. Four pairs of edges which belonged to different squares in Figure 1(a) became identified into four edges, each shared by two squares. To complete folding the cube, one has to identify three more pairs of edges. Each vertex in a three-dimensional cube is adjacent to three squares.



(a) An unfolded 3d cube

(b) The process of folding a cube

(c) An almost folded cube

Figure 1: Folding a 3d cube

The Method of Sections and the Hypercube

Alicia Boole Stott's method of understanding polytopes in four dimensions involved taking sections of them with 3d planes and then studying these sections.

For example, in a 3d cube we can take a section of the cube by a 2-dimensional plane P_1 so that it intersects the cube along one of its faces, for example, along the base square in Figure 1(c). This square corresponds to the central square in Figure 1(a). Now take a plane parallel to P_1, then it intersects four squares in the 3d cube along line segments, which form again a square. In Figure 1(c), this section is represented by a blue rubber band wrapped around the 3d cube.

A polytope in four dimensions, bounded by three-dimensional cubes is called a *hypercube*. Each vertex in the hypercube is adjacent to four 3-dimensional cubes. How many 3-dimensional cubes does one need to build a hypercube?

Figure 2 represents four cubes, adjacent to the same vertex in a hypercube, unfolded into the 3-dimensional space. Just like in Figure 1(a) a pair of edges in the unfolded cube represents the same edge in the folded cube, in Figure 2 a pair of squares in different 3d cubes may represent the same square in the folded hypercube.

To help understand, how cubes are folded together into a four dimensional object, the vertices which fold into the same vertex are marked by the same color. For example, when the hypercube is folded, three vertices in Figure 2, marked by yellow color, are identified into the same vertex. Two vertices, marked by red color are folded into one vertex, two vertices, marked by purple color are folded into one vertex and so on.

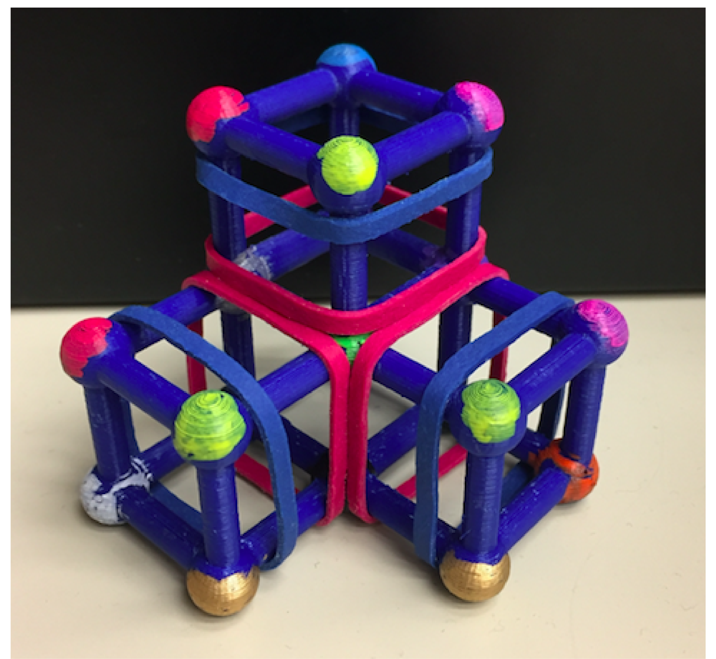


Figure 2: Sections in a hypercube

Let S_1 be a three-dimensional hyperplane in the four-dimensional space, which intersects the hypercube along one of the bounding 3d cubes. In Figure 2, this cube is the central cube which is behind the three cubes which can be seen in the picture. Then take a hyperplane S_2 which is parallel to the hyperplane S_1, and suppose it intersects one of the cubes in Figure 2, except the central one. Then S_2 has to intersect this cube along a square, marked in Figure 2 by a red rubber band. Since the sides of cubes are identified, S_2 also has to interest the other two cubes in Figure 2 along squares. These squares fold into three sides of a 3-dimensional cube. The remaining three squares of the intersection of S_2 and the hypercube are in the cubes which are not represented in Figure 2.

These are the cubes attached to the central cube along the three edges not identified with other cubes in the picture. We conclude that the intersection of S_2 and the hypercube is a 3-dimensional cube.

Let S_3 be a three-dimensional hyperplane, parallel to S_1 and S_2, placed further away from the central cube. Then it intersects the hypercube along a 3-dimensional cube. The intersection is marked by blue rubber bands in Figure 2.

The 8-cell

So far we have counted 7 cubes in the hypercube. These are the central cube in Figure 2, and six cubes attached to the central cube along the squares which form its sides.

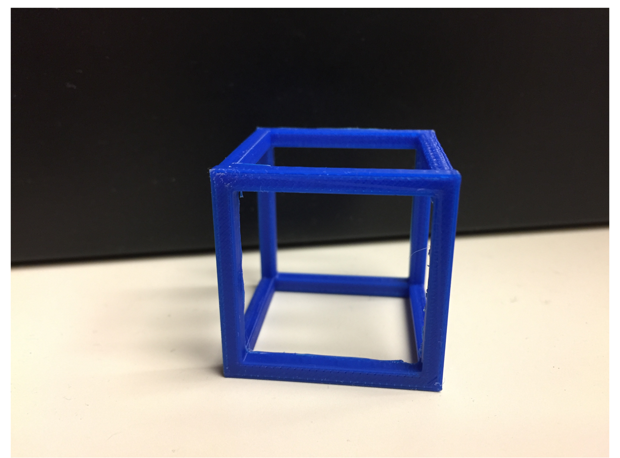


Figure 3: A typical section

Three of the cubes, attached to the sides of the central cube are represented in Figure 2, and three are missing. Sections of the hypercube by three-dimensional hyperplanes S_2 and S_3 are represented by rubber bands of different colors. Each plane intersects a cube adjacent to the central cube in a square, and six squares (three of them are represented in Figure 2, and three are missing) fold into a three-dimensional cube, represented in Figure 3.

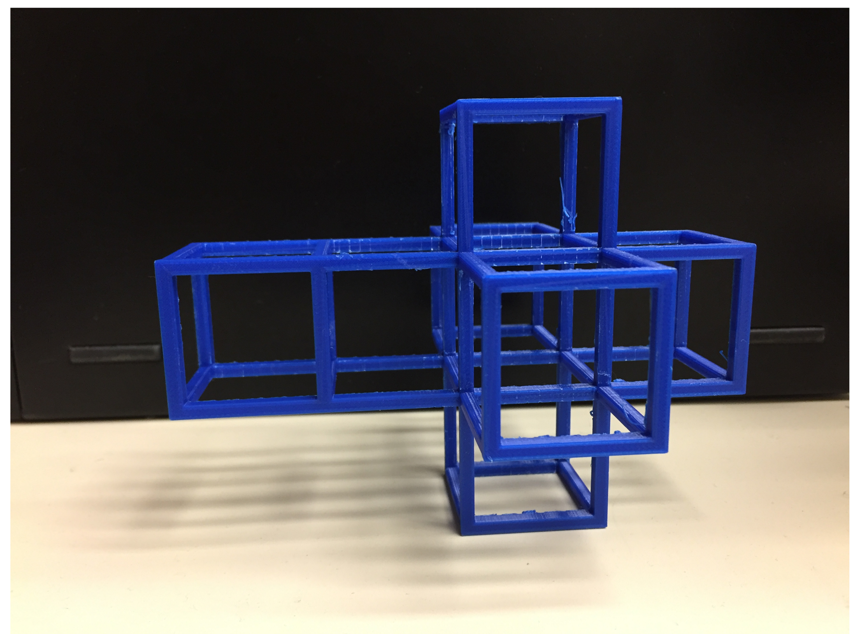


Figure 4: An unfolded 4d cube

If we keep intersecting the hypercube with hyperplanes, eventually a hyperplane S_n will intersect one of the cubes, adjacent to the central cube in Figure 2, in a square which forms one of the sides of this cube. Then S_n will intersect the other 5 cubes along squares, which form their sides. These six squares fold into a cube which is one of the bounding solids in the hypercube.

We see that to build a hypercube in the four-dimensional space we need eight cubes. This is the origin of another name for the hypercube, the 8-cell. An unfolding on the 8-cell into the three-dimensional space is represented in Figure 4.

References

Alicia Boole Stott. *On certain Series of sections of the Regular Four-dimensional Hypersolids*. Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam. **7** (1900), no 3, 1–21.

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