Abel's Theorem Animated Spring 2019

Veronica Kalicki Juan Morales Ryan Ostrander Prof. Alexander Furman (Supervisor)
University of Illinois at Chicago
Mathematical Computing Laboratory

Summary

The project is on *Arnold's* proof of *Abel's* theorem in which he provided a geometric explanation of the theorem. The proof demonstrates that we cannot construct a closed formula that produces the roots of general fifth degree polynomial using a finite combination of field operations, radicals, and elementary functions.

Our goal is to showcase the most important aspects of *Arnold's* proof. Using javascript we developed an animated webpage application that allows users to visually understand the main argument of *Arnold's* proof. Specifically, it shows that given any expression $f : \{a_0, \dots, a_4\} \to \mathbb{C}^5$ that uses elementary functions and radicals, one can construct a *closed path* in the space $\mathrm{Poly}_5(\mathbb{C})$ of monic fifth degree polynomials, such that all values of f return to their original positions, while the roots $z_1, \dots, z_5$ undergo a non-trivial permutation; therefore such f cannot reconstruct the roots $z_1, \dots, z_5$ from the coefficients $a_0, \dots, a_4$.

Motivation

Abel's Theorem asserts *impossibility* of finding a closed formula for the roots of a general polynomial of degree five or higher. The purpose of this animation is to capture the key idea of the proof of the theorem.

Necessity of Radicals for Solving Quadratic Equations

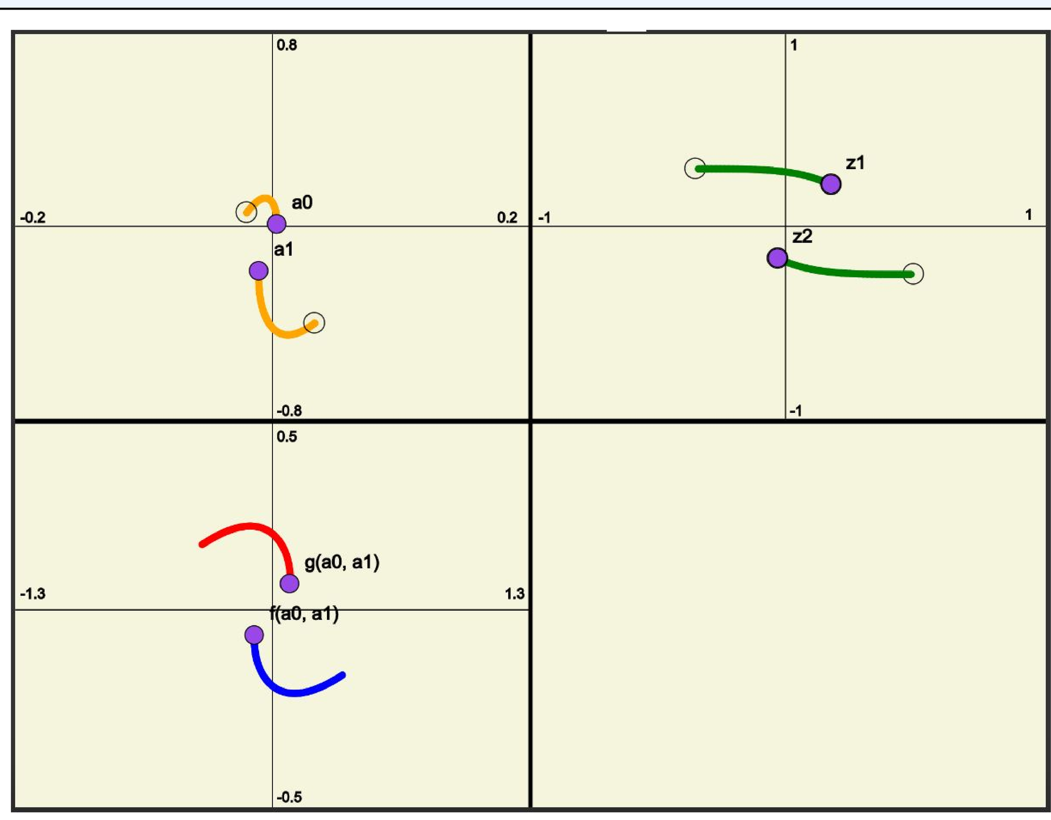

Figure: 1. Case for Monic $p(z)$ in $\mathrm{Poly}_2(\mathbb{C})$

We now show that there is no formula for the roots z_1, z_2 of a general Monic polynomial $p \in \mathrm{Poly}_2(\mathbb{C})$ in terms of analytic (single valued) functions $f, g : \{a_0, a_1\} \to \mathbb{C}$ such that $f(a_0, a_1) = z_1$ and $g(a_0, a_1) = z_2$ for a general quadratic equation.

Suppose otherwise. Then, using *Vieta's* formula we find that the coefficients a_0, a_1 given by $a_0 = z_1 z_2$ and $a_1 = -(z_1 + z_2)$, which are symmetric expressions in z_1, z_2. Starting from distinct points z_1, z_2 we can continuously move them until they change places $z_1 \to z_2$, $z_2 \to z_1$. Under this motion:

- Each of the coefficients a_0, a_1 follows a closed path,
- The functions $f(a_0, a_1)$ and $g(a_0, a_1)$ follow closed paths.

Contradicting the assumption that f and g follow the roots z_1, z_2, that interchanged places.

Conclusion: Any formula would require use of **multi-valued** function (*Quadratic Formula*).

Radicals in Complex Variables

Recall that for any non-zero $z \in \mathbb{C}$ and $n \in \mathbb{N}$ there are precisely n complex numbers w with $w^n = z$.

$$z = r \cdot e^{i\theta}, \qquad w = \sqrt[n]{r} \cdot e^{\frac{i}{n}(\theta + 2k\pi)} \qquad k = 0, 1, \dots, n-1.$$

Let $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ be a closed path starting and ending at z. Then there are precisely n paths $\omega_k : [a, b] \to \mathbb{C} \setminus \{0\}$ that trace the nth roots of $\gamma(t)$.

$$w_k(t)^n = \gamma(t) \qquad t \in [a, b], \quad k = 0, 1, \dots, n-1.$$

Note that while γ is closed $\gamma(a) = z = \gamma(b)$ the paths ω_k need not be closed, yet the map

$$\omega_0(a), \dots, \omega_{n-1}(a) \ \mapsto \ \omega_0(b), \dots, \omega_{n-1}(b)$$

is always a **cyclic permutation** of the n roots of the base point z.

Radical, Functions, and a Commutator

Let $z = f(a_0, a_1, a_2, a_3, a_4)$ be an analytic function in complex variables $a_0, \dots, a_4$, and suppose that for each $j = 0, \dots, 4$ we have two closed loops

$$\beta_j : [0, 1] \to \mathbb{C}, \qquad \gamma_j : [0, 1] \to \mathbb{C}$$

that start and end at some fixed a_j, and such that $f \circ \beta_.$ and $f \circ \gamma_.$ avoid 0. Perform the path

$$[\beta, \gamma] = \beta\gamma\beta^{-1}\gamma^{-1}; \quad \text{(Commutator)}$$

on $a_0, \dots, a_4$ and follow the 5 paths that trace the values of

$$\sqrt[5]{f(a_0, \dots, a_4)}.$$

These paths are **closed loops** because both β and γ define cyclic permutation of the 5 radicals, and cyclic permutations commute.

Arnold/Abel Argument in the Simplest Case

We now rule out the possibility that roots of a Monic polynomial $p \in \mathrm{Poly}_5(\mathbb{C})$ could be expressed by a formula

$$z = g\left(\sqrt[n_1]{f_1(a_0, \dots, a_4)}, \sqrt[n_2]{f_2(a_0, \dots, a_4)}, \dots, \sqrt[n_k]{f_k(a_0, \dots, a_4)} \right) \quad (1)$$

for some analytic $f_1, \dots, f_k : \mathbb{C}^5 \to \mathbb{C}$ and $g : \mathbb{C}^k \to \mathbb{C}$.
Fix distinct $z_1, \dots, z_5$ in $\mathbb{C}$ that represent roots of $p(z)$. Construct continuous paths that move

$$\hat{\beta} : (z_1, z_2, z_3, z_4, z_5) \mapsto (z_2, z_3, z_1, z_4, z_5)$$

and

$$\hat{\gamma} : (z_1, z_2, z_3, z_4, z_5) \mapsto (z_1, z_2, z_4, z_5, z_3)$$

and denote by β, γ the corresponding motions of the coefficients $a_0, \dots, a_4$ of $p(z)$. Then

- Since $\hat{\beta}$ permutes the roots, β_j follows a closed loop (Vieta).
- Each of $f_1(), \dots, f_k()$ follow a closed loop under this motion.
- The paths of $\sqrt[n_i]{f_i}$ amount to a cyclic permutation.
- The same applies to $\hat{\gamma}$.
- Following $\hat{\beta}\hat{\gamma}\hat{\beta}^{-1}\hat{\gamma}^{-1}$ each of the paths of $\sqrt[n_i]{f_i}$ closes up.
- Therefore $g\left(\sqrt[n_1]{f_1}, \dots, \sqrt[n_k]{f_k} \right)$ follows a closed loop.
- But the roots $z_1, \dots, z_5$ got permutated.

So $g\left(\sqrt[n_1]{f_1}, \dots, \sqrt[n_k]{f_k} \right)$ could not follow the roots $z_1, \dots, z_5$

Commutators in S_n

Our ability to solve algebraic equations using radicals is dependent on the solubility of special classes of groups. S_n is the group of all permutations. A_n is the group of all even permutations, where A_n forms a subgroup of S_n.

General Facts:

- i If H is a subgroup of a soluble group G, then H is soluble.
- ii If a group G is not commutative and the only subgroups are the unit element and G itself, then G is not soluble.
- iii For $n \geq 5$, S_n contains a subgroup isomorphic to A_5.
- iv Let G be a finite group. Then G is soluble if and only if there exists $n \in \mathbb{Z}$ such that $G^{(n)} = \{1\}$ (The *nth* commutator group).

Conclusions:

- S_2 is a commutative group and thus soluble.
- S_3 soluble group as $[[a, b], [c, d]] = \{1\}$.
- S_4 soluble group as $[[[a, b], [c, d]], [[a, b], [c, d]]] = \{1\}$.
- By (i) and (iii) it follows that for $n \geq 5$, S_n is not soluble.

Necessity of Nested Radicals to Solve Cubic Equations

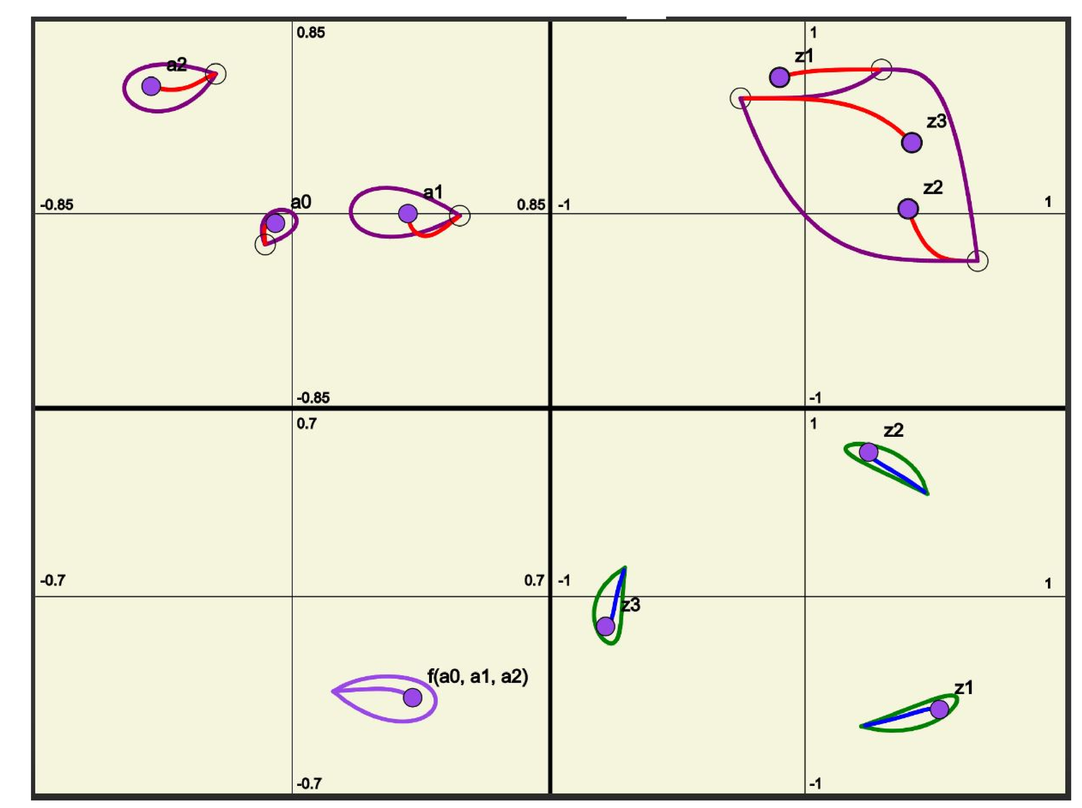

Figure: 2. Case for Monic $p(z)$ in $\mathrm{Poly}_3(\mathbb{C})$

We now demonstrate that a formula for the roots of a Monic polynomial $p \in \mathrm{Poly}_3(\mathbb{C})$ cannot be expressed by a formula $f : \{a_0, a_1, a_2\} \to \mathbb{C}^3$, for f analytic and no nested radicals.

Suppose otherwise, and consider a general Monic polynomial $p \in \mathrm{Poly}_3(\mathbb{C})$ with roots z_1, z_2, z_3. As in (1), no such formula can express the roots of p. We can further construct paths that cyclically permute the roots z_1, z_2, z_3 while inducing closed loops for each coefficient of p and for the value of f. The paths of $\sqrt[n]{f}$ will either follow a closed loop or rotate by some angle θ. Applying the commutator the values of $\sqrt[n]{f}$ undergo a rotation $\Delta\theta = 0$, while inducing a non-trivial permutation of the roots z_1, z_2, z_3, contradiction.
Conclusion: Any formula would require use of **nested radicals** (*Cardano's Formula*).

Arnold's Theorem

Theorem. The Monodromy of the algebraic function $x(a)$ defined by the quintic equation $x^5 + ax + 1 = 0$ is the non-soluble group of the 120 permutations of 5 roots. That is, no function having the same topological branching type as $x(a)$ is representable as a finite combination elementary functions and radicals [1].

Impossibility of Solving the Quintic in Radicals

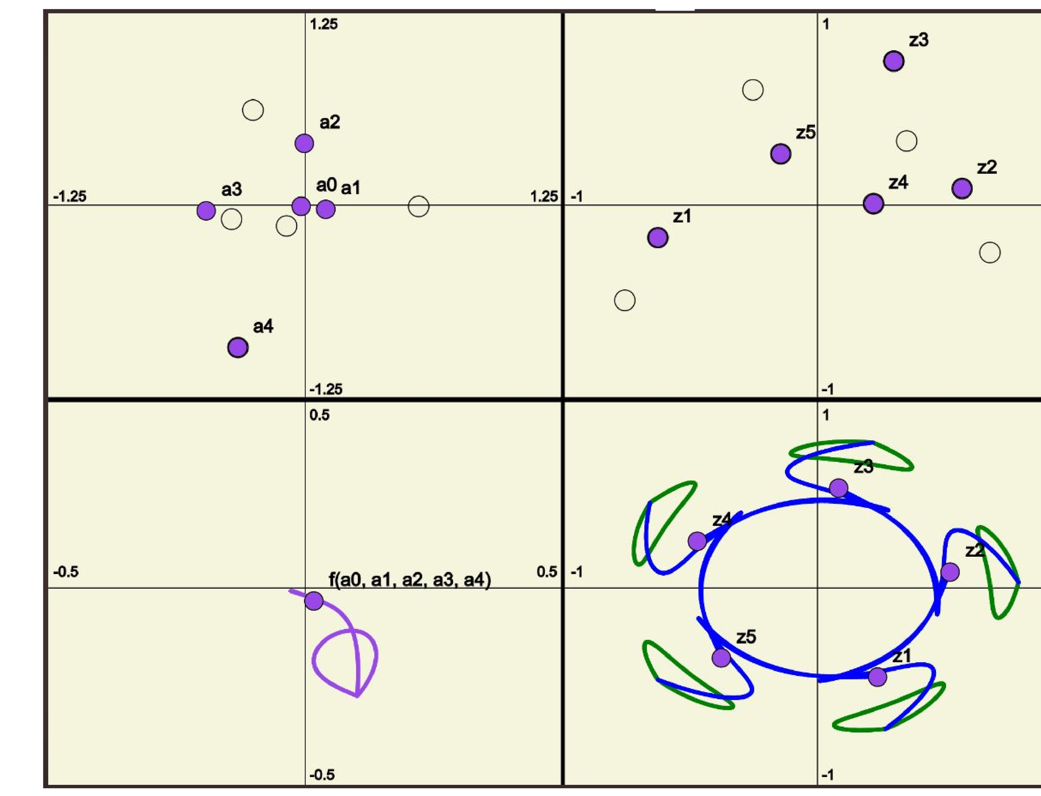

Figure: 3. Case for Monic $p(z)$ in $\mathrm{Poly}_5(\mathbb{C})$

Suppose otherwise, and let $A = \{a_1, \dots a_4\}$. Then, there exist functions $f_1, \dots, f_k : A \to \mathbb{C}^5$ that expresses the roots of a general Monic polynomial $p \in \mathrm{Poly}_5(\mathbb{C})$. By previous case there must be at least $N \geq 3$ levels of nested radicals. Thus, suppose we have an expression of $N \in \mathbb{N}$ levels of nested roots. Since S_5 is not soluble, then we know that there exists continuous paths such that their commutator induce a non-trivial permutation of the roots, while both the coefficients of p and f_i, for $1 \leq i \leq k$, follow a **closed loop**, contradiction.
Conclusion: One cannot construct an expression for the roots of a general $p \in \mathrm{Poly}_5(\mathbb{C})$ in terms of it's coefficients, analytic functions, and radicals.

Challenges

One of our biggest challenges was a matter of translating abstract mathematical concepts into explicit computer language that allowed us to obtain the results we desired.

Conclusion

Our project provides a comprehensive illustration of *Arnold's* proof of *Abel's* theorem in an accessible manner. Our goal was to illustrate these abstract mathematical concepts into a language that anyone with basic knowledge of mathematics can understand and appreciate, while building intuition of the mathematics occurring in the background.

References

V. B. Alekseev.
Abel's theorem in problems and solutions.
Kluwer Academic Publishers, Dordrecht, 2004.
Based on the lectures of Professor V. I. Arnold, With a preface and an appendix by Arnold and an appendix by A. Khovanskii.